A GENERALISATION OF THE CASSELS AND GREUB-REINBOLDT INEQUALITIES IN INNER PRODUCT SPACES

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Abstract. A generalisation of the Cassels and Greub-Reinboldt inequalities in complex or real inner product spaces and applications for isotonic linear functionals, integrals and sequences are provided.

1. Introduction

The following result was proved by J.W.S. Cassels in 1951 (see Appendix 1 of [10]).

Theorem 1. *Let* $\overline{\mathbf{a}} = (a_1, \dots, a_n)$, $\overline{\mathbf{b}} = (b_1, \dots, b_n)$ *be sequences of positive real numbers and* $\overline{\mathbf{w}} = (w_1, \dots, w_n)$ *a sequence of nonnegative real numbers. Suppose that*

$$m = \min_{i=\overline{1,n}} \left\{ \frac{a_i}{b_i} \right\} \quad \text{and} \quad M = \max_{i=\overline{1,n}} \left\{ \frac{a_i}{b_i} \right\}. \tag{1.1}$$

Then one has the inequality

$$\frac{\sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i b_i^2}{\left(\sum_{i=1}^n w_i a_i b_i\right)^2} \leq \frac{(m+M)^2}{4mM}. \tag{1.2}$$

The equality holds in (1.2) when $w_1 = \frac{1}{a_1 b_1}$, $w_n = \frac{1}{a_n b_n}$, $w_2 = \dots = w_{n-1} = 0$, $m = \frac{a_n}{b_1}$ *and* $M = \frac{a_1}{b_n}$.

If one assumes that $0 < a \leq a_i \leq A < \infty$ and $0 < b \leq b_i \leq B < \infty$ for each $i \in \{1, \dots, n\}$, then by (2.2) we may obtain Greub-Reinboldt's inequality [3]

$$\frac{\sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i b_i^2}{\left(\sum_{i=1}^n w_i a_i b_i\right)^2} \leq \frac{(ab+AB)^2}{4abAB}. \tag{1.3}$$

The following "unweighted" Cassels' inequality also holds

$$\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2}{\left(\sum_{i=1}^n a_i b_i\right)^2} \leq \frac{(m+M)^2}{4mM}, \tag{1.4}$$

provided $\overline{\mathbf{a}}$ and $\overline{\mathbf{b}}$ satisfy (1.1). This inequality will produce the well known Pólya-Szegö inequality [7, pp. 57, 213-114], [4, pp. 71-72, 253-255]:

$$\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2}{\left(\sum_{i=1}^n a_i b_i\right)^2} \leq \frac{(ab+AB)^2}{4abAB}, \tag{1.5}$$

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provided $0 < a \leq a_i \leq A < \infty$ and $0 < b \leq b_i \leq B < \infty$ for each $i \in \{1, \dots, n\}$.

In [6], C.P. Niculescu proved, amongst others, the following generalisation of Cassels' inequality:

Theorem 2. *Let E be a vector space endowed with a Hermitian product $\langle \cdot, \cdot \rangle$. Then*

$$\frac{\operatorname{Re}\langle x, y\rangle}{\langle x, x\rangle^{\frac{1}{2}} \langle y, y\rangle^{\frac{1}{2}}} \geq \frac{2}{\sqrt{\frac{\omega}{\Omega}} + \sqrt{\frac{\Omega}{\omega}}} \tag{1.6}$$

for every $x, y \in E$ and every $\omega, \Omega > 0$ for which $\operatorname{Re}\langle x - \omega y, x - \Omega y\rangle \leq 0$.

For other reverses of the Cauchy-Bunyakovsky-Schwarz, see the references [1]-[10].

In this paper we obtain a generalisation of (1.6) for complex numbers ω and Ω for which $\operatorname{Re}(\overline{\omega}\Omega) > 0$. Applications for isotonic linear functionals, integrals and sequences are also given.

2. The Results

The following reverse of Schwarz's inequality in inner product spaces holds.

Theorem 3. *Let $a, A \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ so that $\operatorname{Re}(\overline{a}A) > 0$. If $x, y \in H$ are such that*

$$\operatorname{Re}\langle Ay - x, x - ay\rangle \geq 0, \tag{2.1}$$

then one has the inequality

$$\|x\| \|y\| \leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right]}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \leq \frac{1}{2} \cdot \frac{|A| + |a|}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} |\langle x, y\rangle|. \tag{2.2}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.

Proof. We have, obviously, that

$$I := \operatorname{Re}\langle Ay - x, x - ay\rangle = \operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right] - \|x\|^2 - [\operatorname{Re}(\overline{a}A)] \|y\|^2$$

and, thus, by (2.1), one has

$$\|x\|^2 + [\operatorname{Re}(\overline{a}A)] \cdot \|y\|^2 \leq \operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right],$$

giving

$$\frac{1}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \|x\|^2 + [\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}} \|y\|^2 \leq \frac{\operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right]}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}}. \tag{2.3}$$

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq$$

holding for $p, q \geq 0$ and $\alpha > 0$, we deduce

$$2\|x\| \|y\| \leq \frac{1}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \|x\|^2 + [\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}} \|y\|^2. \tag{2.4}$$

Utilizing (2.3) and (2.4) we deduce the first part of (2.2).

The last part is obvious by the fact that for $z \in \mathbb{C}$, $|\operatorname{Re}(z)| \leq |z|$.

Now, assume that the first inequality in (2.2) holds with a constant $c > 0$, i.e.,

$$\|x\| \|y\| \leq c \frac{\operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right]}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}}, \tag{2.5}$$

where a, A, x and y satisfy (2.2).

If we choose $a = A = 1$, $y = x \neq 0$, then obviously (2.2) holds and from (2.5) we may obtain

$$\|x\|^2 \leq 2c \|x\|^2,$$

giving $c \geq \frac{1}{2}$.

The theorem is completely proved. ∎

The following corollary is a natural consequence of the above theorem.

Corollary 1. *Let $m, M > 0$. If $x, y \in H$ are such that*

$$\operatorname{Re}\langle My - x, x - my\rangle \geq 0, \tag{2.6}$$

then one has the inequality

$$\|x\| \|y\| \leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} \operatorname{Re}\langle x, y\rangle \leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} |\langle x, y\rangle|. \tag{2.7}$$

The constant $\frac{1}{2}$ is sharp in (2.7).

Remark 1. *The inequality (2.7) is equivalent to Niculescu's inequality (1.6).*

The following corollary is also obvious.

Corollary 2. *With the assumptions of Corollary 1, we have*

$$\begin{aligned} 0 &\leq \|x\| \|y\| - |\langle x, y\rangle| \leq \|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \\ &\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\langle x, y\rangle \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} |\langle x, y\rangle| \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} 0 &\leq \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \|x\|^2 \|y\|^2 - [\operatorname{Re}\langle x, y\rangle]^2 \\ &\leq \frac{(M-m)^2}{4mM} [\operatorname{Re}\langle x, y\rangle]^2 \leq \frac{(M-m)^2}{4mM} |\langle x, y\rangle|^2. \end{aligned} \tag{2.9}$$

Proof. If we subtract $\operatorname{Re}\langle x, y\rangle \geq 0$ from the first inequality in (2.7), we get

$$\begin{aligned} \|x\| \|y\| - \operatorname{Re}\langle x, y\rangle &\leq \left(\frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} - 1\right) \operatorname{Re}\langle x, y\rangle \\ &= \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\langle x, y\rangle \end{aligned}$$

which proves the third inequality in (2.8). The other ones are obvious.

Now, if we square the first inequality in (2.7) and then subtract $[\operatorname{Re}\langle x, y\rangle]^2$, we get

$$\begin{aligned} \|x\|^2 \|y\|^2 - [\operatorname{Re}\langle x, y\rangle]^2 &\leq \left[\frac{(M+m)^2}{4mM} - 1\right] [\operatorname{Re}\langle x, y\rangle]^2 \\ &= \frac{(M-m)^2}{4mM} [\operatorname{Re}\langle x, y\rangle]^2 \end{aligned}$$

which proves the third inequality in (2.5). The other ones are obvious. ∎

3. Applications for Isotonic Linear Functionals

Let $F(T)$ be an algebra of real functions defined on T and L a subclass of $F(T)$ satisfying the conditions:

(i) $f, g \in L$ implies $f + g \in L$;
(ii) $f \in L, \in \mathbb{R}$ implies $\alpha f \in L$.

A functional A defined on L is an *isotonic linear functional* on L provided that

(a) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in L$;
(aa) $f \geq g$, that is, $f(t) \geq g(t)$ for all $t \in T$, implies $A(f) \geq A(g)$.

The functional A is *normalised* on L, provided that $\mathbf{1} \in L$, i.e., $\mathbf{1}(t) = 1$ for all $t \in T$, implies $A(\mathbf{1}) = 1$.

Usual examples of isotonic linear functionals are integrals, sums, etc.

Now, suppose that $h \in F(T)$, $h \geq 0$ is given and satisfies the properties that $fgh \in L$, $fh \in L$, $gh \in L$ for all $f, g \in L$. For a given isotonic linear functional $A : L \to \mathbb{R}$ with $A(h) > 0$, define the mapping $(\cdot, \cdot)_{A,h} : L \times L \to \mathbb{R}$ by

$$(f, g)_{A,h} := \frac{A(fgh)}{A(h)}.$$

This functional satisfies the following properties:

(s) $(f, f)_{A,h} \geq 0$ for all $f \in L$;
(ss) $(\alpha f + \beta g, k)_{A,h} = \alpha (f, k)_{A,h} + \beta (g, k)_{A,h}$ for all $f, g, k \in L$ and $\alpha, \beta \in \mathbb{R}$;
(sss) $(f, g)_{A,h} = (g, f)_{A,h}$ for all $f, g \in L$.

The following proposition holds.

Proposition 1. *Let $f, g, h \in F(T)$ be such that $fgh \in L$, $f^2h \in L$, $g^2h \in L$. If $m, M > 0$ are such that*

$$mg \leq f \leq Mg \text{ on } F(T), \tag{3.1}$$

then for any isotonic linear functional $A : L \to \mathbb{R}$ with $A(h) > 0$, we have the inequality

$$1 \leq \frac{A(f^2h) A(g^2h)}{A^2(fgh)} \leq \frac{(M+m)^2}{4mM}. \tag{3.2}$$

The constant $\frac{1}{4}$ in (3.2) is sharp.

Proof. We observe that

$$(Mg - f, f - mg)_{A,h} = A[h(Mg - f)(f - mg)] \geq 0.$$

Applying Corollary 1 for $(\cdot, \cdot)_{A,L}$ we get

$$1 \leq \frac{(f, f)_{A,h} (g, g)_{A,h}}{(f, g)^2_{A,h}} \leq \frac{(M+m)^2}{4mM},$$

which is clearly equivalent to (3.2). ∎

The following additive versions of (3.2) also hold.

Corollary 3. *With the assumption in Proposition 1, one has*

$$0 \le \left[A\left(f^2 h\right) A\left(g^2 h\right)\right]^{\frac{1}{2}} - A(hfg) \le \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} A(hfg) \tag{3.3}$$

and

$$0 \le A\left(f^2 h\right) A\left(g^2 h\right) - A^2(fgh) \tag{3.4}$$

$$\le \frac{(M-m)^2}{4mM} A^2(fgh). \tag{3.5}$$

Remark 2. *The condition (3.1) may be replaced with the weaker assumption*

$$(Mg - f, f - mg)_{A,h} \ge 0. \tag{3.6}$$

Remark 3. *With the assumption (3.1) or (3.6) and if $f, g \in F(T)$ with $fg, f^2, g^2 \in L$, then one has the inequality*

$$1 \le \frac{A\left(f^2\right) A\left(g^2\right)}{A^2(fg)} \le \frac{(M+m)^2}{4mM}, \tag{3.7}$$

$$0 \le \left[A\left(f^2\right) A\left(g^2\right)\right]^{\frac{1}{2}} - A(fg) \le \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} A(fg) \tag{3.8}$$

and

$$0 \le A\left(f^2\right) A\left(g^2\right) - A^2(fg) \le \frac{(M-m)^2}{4mM} A^2(fg). \tag{3.9}$$

4. Applications for Integrals

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a $\sigma-$algebra Σ of subsets of Ω and a countably additive and positive measure on Σ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L^2_\rho(\Omega, \mathbb{K})$ the Hilbert space of all $\mathbb{K}$-valued functions f defined on Ω that are $2-\rho-$integrable on Ω, i.e., $\int_\Omega \rho(t) |f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \to [0, \infty)$ is a measurable function on Ω.

The following proposition contains a counterpart of the weighted Cauchy-Bunyakovsky-Schwarz's integral inequality.

Proposition 2. *Let $A, a \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ with $\operatorname{Re}(\overline{a}A) > 0$ and $f, g \in L^2_\rho(\Omega, \mathbb{K})$. If*

$$\int_\Omega \operatorname{Re}\left[(Ag(s) - f(s))\left(\overline{f(s)} - \overline{a}\,\overline{g}(s)\right)\right] \rho(s)\, d\mu(s) \ge 0, \tag{4.1}$$

then one has the inequality

$$\left[\int_{\Omega}\rho(s)|f(s)|^2 d\mu(s)\int_{\Omega}\rho(s)|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} \tag{4.2}$$
$$\leq \frac{1}{2}\cdot\frac{\int_{\Omega}\rho(s)\operatorname{Re}\left[A\overline{f(s)}g(s)+\overline{a}f(s)\overline{g(s)}\right]d\mu(s)}{\left[\operatorname{Re}(\overline{a}A)\right]^{\frac{1}{2}}}$$
$$\leq \frac{1}{2}\cdot\frac{|A|+|a|}{\left[\operatorname{Re}(\overline{a}A)\right]^{\frac{1}{2}}}\left|\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right|.$$

The constant $\frac{1}{2}$ is sharp in (4.2).

Proof. Follows by Theorem 3 applied for the inner product $\langle\cdot,\cdot\rangle_\rho := L^2_\rho(\Omega,\mathbb{K})\times L^2_\rho(\Omega,\mathbb{K})\to\mathbb{K}$,

$$\langle f,g\rangle := \int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s).$$

∎

Remark 4. *A sufficient condition for (4.1) to hold is*

$$\operatorname{Re}\left[(Ag(s)-f(s))\left(\overline{f(s)}-\overline{a}\overline{g(s)}\right)\right]\geq 0 \quad \textit{for } \mu\textit{-a.e.} \quad s\in\Omega. \tag{4.3}$$

In the particular case $\rho = 1$, we have the following result.

Corollary 4. *Let $a, A \in \mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ with $\operatorname{Re}(\overline{a}A)>0$ and $f,g\in L^2(\Omega,\mathbb{K})$. If*

$$\int_{\Omega}\operatorname{Re}\left[(Ag(s)-f(s))\left(\overline{f(s)}-\overline{a}\overline{g(s)}\right)\right]d\mu(s)\geq 0, \tag{4.4}$$

then one has the inequality

$$\left[\int_{\Omega}|f(s)|^2 d\mu(s)\int_{\Omega}|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} \tag{4.5}$$
$$\leq \frac{1}{2}\cdot\frac{\int_{\Omega}\operatorname{Re}\left[A\overline{f(s)}g(s)+\overline{a}f(s)\overline{g(s)}\right]d\mu(s)}{\left[\operatorname{Re}(\overline{a}A)\right]^{\frac{1}{2}}}$$
$$\leq \frac{1}{2}\cdot\frac{|A|+|a|}{\left[\operatorname{Re}(\overline{a}A)\right]^{\frac{1}{2}}}\left|\int_{\Omega}f(s)\overline{g(s)}d\mu(s)\right|.$$

Remark 5. *If $\mathbb{K}=\mathbb{R}$, then a sufficient condition for either (4.1) or (4.4) to hold is*

$$ag(s)\leq f(s)\leq Ag(s) \quad \textit{for } \mu\textit{-a.e. } s\in\Omega, \tag{4.6}$$

where, in this case, $a, A\in\mathbb{R}$ with $A>a>0$.

When a, A are real positive constants, then the following proposition holds.

Proposition 3. *Let $m, M>0$. If $f,g\in L^2_\rho(\Omega,\mathbb{K})$ such that*

$$\int_{\Omega}\rho(s)\operatorname{Re}\left[(Mg(s)-f(s))\left(\overline{f(s)}-m\overline{g}(s)\right)\right]d\mu(s)\geq 0 \tag{4.7}$$

then one has the inequality

$$\begin{aligned}(4.8)\quad & \left[\int_{\Omega}\rho(s)\left|f(s)\right|^{2}d\mu(s)\int_{\Omega}\rho(s)\left|g(s)\right|^{2}d\mu(s)\right]^{\frac{1}{2}} \\ & \qquad\leq\frac{1}{2}\cdot\frac{M+m}{\sqrt{mM}}\int_{\Omega}\rho(s)\operatorname{Re}\left[f(s)\overline{g(s)}\right]d\mu(s).\end{aligned}$$

The proof follows by Corollary 1 applied for the inner product

$$\langle f,g\rangle_{\rho}:=\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s).$$

The following additive versions also hold.

Corollary 5. *With the assumptions in Proposition 3, one has the inequalities*

$$\begin{aligned}(4.9)\qquad 0 & \leq\left[\int_{\Omega}\rho(s)\left|f(s)\right|^{2}d\mu(s)\int_{\Omega}\rho(s)\left|g(s)\right|^{2}d\mu(s)\right]^{\frac{1}{2}} \\ & \qquad -\int_{\Omega}\rho(s)\operatorname{Re}\left[f(s)\overline{g(s)}\right]d\mu(s) \\ & \leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^{2}}{2\sqrt{mM}}\int_{\Omega}\rho(s)\operatorname{Re}\left[f(s)\overline{g(s)}\right]d\mu(s)\end{aligned}$$

and

$$\begin{aligned}(4.10)\qquad 0 & \leq\int_{\Omega}\rho(s)\left|f(s)\right|^{2}d\mu(s)\int_{\Omega}\rho(s)\left|g(s)\right|^{2}d\mu(s) \\ & \qquad -\left(\int_{\Omega}\rho(s)\operatorname{Re}\left[f(s)\overline{g(s)}\right]d\mu(s)\right)^{2} \\ & \leq\frac{(M-m)^{2}}{4mM}\left(\int_{\Omega}\rho(s)\operatorname{Re}\left[f(s)\overline{g(s)}\right]d\mu(s)\right)^{2}.\end{aligned}$$

Remark 6. *If $\mathbb{K}=\mathbb{R}$, a sufficient condition for (4.1) to hold is*

$$(4.11)\qquad mg(s)\leq f(s)\leq Mg(s)\quad\text{for}\quad \mu\text{-a.e. } s\in\Omega,$$

where $M>m>0$.

5. Applications for Sequences

For a given sequence $(w_i)_{i\in\mathbb{N}}$ of nonnegative real numbers, consider the Hilbert space $\ell_w^2(\mathbb{K})$, $(\mathbb{K}=\mathbb{C},\mathbb{R})$, where

$$(5.1)\qquad \ell_w^2(\mathbb{K}):=\left\{\overline{x}=(x_i)_{i\in\mathbb{N}}\subset\mathbb{K}\,\middle|\,\sum_{i=0}^{\infty}w_i\left|x_i\right|^2<\infty\right\}.$$

The following proposition that provides a counterpart of the weighted Cauchy-Bunyakovsky-Schwarz inequality for complex numbers holds.

Proposition 4. *Let $a,A\in\mathbb{K}$ with $\operatorname{Re}(\overline{a}A)>0$ and $\overline{x},\overline{y}\in\ell_w^2(\mathbb{K})$. If*

$$(5.2)\qquad \sum_{i=0}^{\infty}w_i\operatorname{Re}\left[(Ay_i-x_i)\left(\overline{x_i}-\overline{ay_i}\right)\right]\geq 0,$$

then one has the inequality

$$\left[\sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2\right]^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{\sum_{i=0}^{\infty} w_i \operatorname{Re}\left[A\overline{x_i}y_i + \overline{a}x_i\overline{y_i}\right]}{\left[\operatorname{Re}\left(\overline{a}A\right)\right]^{\frac{1}{2}}} \leq \frac{1}{2} \cdot \frac{|A| + |a|}{\left[\operatorname{Re}\left(\overline{a}A\right)\right]^{\frac{1}{2}}} \left|\sum_{i=0}^{\infty} w_i x_i \overline{y_i}\right|. \tag{5.3}$$

The constant $\frac{1}{2}$ is sharp in (5.3).

Proof. Follows by Theorem 3 applied for the inner product $\langle \cdot, \cdot \rangle : \ell_w^2(\mathbb{K}) \times \ell_w^2(\mathbb{K}) \to \mathbb{K}$,

$$\langle \overline{x}, \overline{y} \rangle_w := \sum_{i=0}^{\infty} w_i x_i \overline{y_i}.$$

∎

Remark 7. *A sufficient condition for (5.2) to hold is*

$$\operatorname{Re}\left[(Ay_i - x_i)\left(\overline{x_i} - \overline{a y_i}\right)\right] \geq 0 \quad \text{for all } i \in \mathbb{N}. \tag{5.4}$$

In the particular case $\rho = 1$, we have the following result.

Corollary 6. *Let $a, A \in \mathbb{K}$ with $\operatorname{Re}(\overline{a}A) > 0$ and $\overline{x}, \overline{y} \in \ell^2(\mathbb{K})$. If*

$$\sum_{i=0}^{\infty} \operatorname{Re}\left[(Ay_i - x_i)\left(\overline{x_i} - \overline{a y_i}\right)\right] \geq 0, \tag{5.5}$$

then one has the inequality

$$\left[\sum_{i=0}^{\infty} |x_i|^2 \sum_{i=0}^{\infty} |y_i|^2\right]^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{\sum_{i=0}^{\infty} \operatorname{Re}\left[A\overline{x_i}y_i + \overline{a}x_i\overline{y_i}\right]}{\left[\operatorname{Re}\left(\overline{a}A\right)\right]^{\frac{1}{2}}} \leq \frac{1}{2} \cdot \frac{|A| + |a|}{\left[\operatorname{Re}\left(\overline{a}A\right)\right]^{\frac{1}{2}}} \left|\sum_{i=0}^{\infty} x_i \overline{y_i}\right|. \tag{5.6}$$

Remark 8. *If $\mathbb{K} = \mathbb{R}$, then a sufficient condition for either (5.1) or (5.4) to hold is*

$$a y_i \leq x_i \leq A y_i \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{5.7}$$

where, in this case, $a, A \in \mathbb{R}$ with $aA > 0$.

When the constants are positive, then the following proposition also holds.

Proposition 5. *Let $m, M > 0$. If $\overline{x}, \overline{y} \in \ell_w^2(\mathbb{K})$ such that*

$$\sum_{i=0}^{\infty} w_i \operatorname{Re}\left[(M y_i - x_i)\left(\overline{x_i} - m\overline{y_i}\right)\right] \geq 0, \tag{5.8}$$

then one has the inequality

$$\left[\sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2\right]^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{M + m}{\sqrt{mM}} \sum_{i=0}^{\infty} w_i \operatorname{Re}\left(x_i \overline{y_i}\right). \tag{5.9}$$

The proof follows by Corollary 1 applied for the inner product

$$\langle \overline{x}, \overline{y} \rangle_w := \sum_{i=0}^{\infty} w_i x_i, \overline{y_i}.$$

The following additive version also holds.

Corollary 7. *With the assumptions in Proposition 5, one has the inequalities*

$$0 \le \left[\sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2 \right]^{\frac{1}{2}} - \sum_{i=0}^{\infty} w_i \operatorname{Re}(x_i \overline{y_i}) \le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \sum_{i=0}^{\infty} w_i \operatorname{Re}(x_i \overline{y_i}) \tag{5.10}$$

and

$$0 \le \sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2 - \left[\sum_{i=0}^{\infty} w_i \operatorname{Re}(x_i \overline{y_i}) \right]^2 \le \frac{(M-m)^2}{4mM} \left[\sum_{i=0}^{\infty} w_i \operatorname{Re}(x_i \overline{y_i}) \right]^2. \tag{5.11}$$

Remark 9. *If $\mathbb{K} = \mathbb{R}$, a sufficient condition for (5.8) to hold is*

$$my_i \le x_i \le My_i \quad \text{for each} \quad i \in \mathbb{N}, \tag{5.12}$$

where $M > m > 0$.

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